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Oscillation and nonoscillation of two-dimensional difference systems[☆]

Jianchu Jiang[a, b, *], Xiaoping Li[a]

[a] *Department of Mathematics, Humanities and Science and Technology Institute of Hunan, Loudi, Hunan 417000, China*
[b] *Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore*

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Abstract

This paper is concerned with the oscillation and nonoscillation behavior of solution of the nonlinear two-dimensional difference system

$$\Delta x_n = a_n g(y_n),$$

$$\Delta y_{n-1} = -f(n, x_n), \quad n \in N(n_0) = \{n_0, n_0 + 1, \ldots\}.$$

Some necessary and sufficient conditions are given for the system to admit the existence of oscillatory and nonoscillatory solutions with special asymptotic properties. Some important results in the literatures are generalized. Examples to illustrate the result are included.

MSC: 39A12

Keywords: Nonlinear difference system; Oscillation; Nonoscillation

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* Corresponding author. Department of Mathematics, Humanities and Science and Technology Institute of Hunan, Loudi, Hunan 417000, China. Tel.: +86 738 832 5290; fax: +86 738 832 5700.
E-mail address: jianchujiang@yahoo.com.cn (J. Jiang).

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1. Introduction

Consider the nonlinear two-dimensional difference system

$$\Delta x_n = a_n g(y_n), \quad \Delta y_{n-1} = -f(n, x_n), \quad n \in N(n_0) = \{n_0, n_0+1, \ldots\}, \tag{1}$$

where Δ is defined by $\Delta y_n = y_{n+1} - y_n$, $n \in N(n_0) = \{n_0, n_0+1, \ldots\}$, $n_0 \in N = \{1, 2, \ldots\}$, $\{a_n\}$ is the nonnegative real sequence, $g(u) : R \to R$ is continuous function with properties, $ug(u) > 0$, for $u \neq 0$. $f(n, u) : N(n_0) \times R \to R$ is continuous as a function of $u \in R$; $uf(n, u) > 0$ for $n \in N(n_0)$ and $u \neq 0$.

Throughout the paper, we will restrict our attention to only the solutions of system (1) that exist for $n \in N(N_0)$, where $N_0 \geqslant n_0$. As usual, a real sequence defined on $N(N_0)$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is said to be nonoscillatory otherwise. A solution $(\{x_n\}, \{y_n\})$ of system (1) will be called oscillatory if both components are oscillatory, and it will be called nonoscillatory otherwise. The difference system (1) will be called oscillatory if all its solutions are oscillatory.

The difference system (1) can be considered as a natural generalization of difference system

$$\Delta x_n = a_n g(y_n), \quad \Delta y_{n-1} = -b_n f(x_n), \quad n \in N(n_0) = \{n_0, n_0+1, \ldots\}. \tag{2}$$

We note the relevance of systems of form (2) as several important second-order difference equation

$$\Delta(a_{n-1}\Delta x_{n-1}) + p_n f(x_n) = 0, \tag{3}$$

$$\Delta^2 x_{n-1} + p_n |x_n|^{\lambda} sgn\, x_n = 0 \tag{4}$$

and

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\alpha}) + p_n f(x_n) = 0, \tag{5}$$

can be written in form (2). For oscillation and nonoscillation criteria regarding Eqs. (3)–(5), we refer to [1,3,5,8]. System (2) has been an object of intensive studies in recent years. However, most of previous studies have been restricted to the case where $f(n, y) = b_n f(y_n)$, we refer to [2,4,6,7,9]. Graef [4] obtained some oscillation results, Li [9] obtained some necessary and sufficient conditions of nonoscillation for system (2), and very little is known about the case of general $f(n, y)$ in which n and y are not necessarily separable. Up to now, even for the system

$$\Delta x_n = \frac{1}{n} y_n^4, \quad \Delta y_{n-1} = -\frac{n^3 |x_n|^{4/3} x_n}{2004 + n^3 x_n^2}, \quad n \in N = \{1, 2, \ldots\},$$

the known result cannot be applied to determine the oscillatory and nonoscillatory behaviors. A question arises: Is it possible to establish a necessary and sufficient condition for all solutions of (1) to be oscillatory? Our results answer this question.

The paper is organized as follows. In the next section, we generalize some results of [9], some necessary and sufficient conditions are given for the system to admit the existence of nonoscillatory solutions with special asymptotic properties. In Section 3, we obtain some necessary and sufficient conditions for all solutions of system (1) to be oscillatory, some examples to illustrate our result are given in Section 4.

2. Nonoscillation results

In this section, we generalize some results of [9] to system (1). We are interested in characterizing the situation in which system (1) to admit the existence of nonoscillatory solutions with special asymptotic properties, these results will be used for the next section. Additional hypotheses on $f(t, y)$ and $g(x)$ are needed for this purpose.

(c_1) $g(x)$ is nondecreasing.
(c_2) For any positive constant l and L with $l < L$ there exist positive constants h and H, depending possibly on l and L such that $l \leqslant |y| \leqslant L$ implies

$$hf(n, l) \leqslant |f(n, y)| \leqslant Hf(n, L), \quad n \in N(n_0).$$

(c_3) For any positive constant l and L with $l < L$ there exist positive constants h and H, depending possibly on l and L such that $l \leqslant |y| \leqslant L$ implies

$$hf(n, l\theta(n)) \leqslant |f(n, \theta(n)y)| \leqslant Hf(n, L\theta(n)), \quad n \in N(n_0),$$

where $\theta(n)$ is a positive nonincreasing function.
(c_4) $g(uv) = g(u)g(v)$ for $uv > 0$.
(c_5) $\lim_{n\to\infty} A_{n_0,n} = \infty$, where $A_{m,n} = \sum_{s=m}^{n-1} a_s$.

Theorem 1. *Suppose that* (c_1) *and* (c_2) *are satisfied. Then, system* (1) *has a nonoscillatory solution* $(\{x_n\}, \{y_n\})$ *such that* $\lim_{n\to\infty} x_n = \alpha \neq 0$ *and* $\lim_{n\to\infty} y_n = 0$ *if and only if for any positive constant* d

$$\sum_{n_0}^{\infty} a_n g\left(d \sum_{r=n}^{\infty} |f(r, c)|\right) < \infty \quad \textit{for some } c \neq 0. \tag{6}$$

Proof. Necessity. Let $(\{x_n\}, \{y_n\})$ be a nonoscillatory solution of system (1) such that $\lim_{n\to\infty} x_n = \text{const} \neq 0$ and $\lim_{n\to\infty} y_n = 0$. There is no loss of generality in assuming that $\lim_{n\to\infty} x_n > 0$. Then there exist positive constants $l,\ L$ and $n_1 \in N(n_0)$ such that $l \leqslant x_n \leqslant L$ for $n \geqslant n_1$. Condition (c_2) then implies that

$$f(n, x_n) \geqslant hf(n, l) \quad \text{for } n \geqslant n_1, \tag{7}$$

for some constant $h > 0$.

Summing the second equation of (1) from $n + 1$ to ∞, and noting that $\lim_{n\to\infty} y_n = 0$, we have

$$y_n = \sum_{r=n+1}^{\infty} f(r, x_r), \quad n \geqslant n_1.$$

Thus, summing the first equation of (1), we have

$$\begin{aligned}\infty > \lim_{n\to\infty} x_n - x_{n_1} &= \sum_{m=n_1}^{\infty} a_n g(y_n)\\ &= \sum_{s=n_1}^{n} a_n g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right)\\ &\geqslant \sum_{s=n_1}^{\infty} a_n g\left(h \sum_{s+1}^{\infty} f(r, l)\right),\end{aligned}$$

which implies that (6) holds with $d = h, c = l$.

Sufficiency. Suppose that (6) holds for some $c \neq 0$ and all $d > 0$, we may assume that $c > 0$. By condition (c_2), there is a constant $H > 0$ such that $c/2 \leqslant x_n \leqslant c$ implies

$$f(n, x_n) \leqslant H f(n, c) \quad \text{for } n \in N(n_0).$$

Choose $N_0 \in N(n_0)$ to be large so that

$$\sum_{s=N_0}^{\infty} a_s g\left(H \sum_{r=s+1}^{\infty} f(r, c)\right) \leqslant \frac{c}{2}. \tag{8}$$

Consider the Banach space Φ of all bounded sequence $x = \{x_n\}$ with norm $\|x\| = \sup_{n\in N(n_0)}|x_n|$, we defined a bounded, convex and closed subset X of Φ as

$$X = \left\{x \in \Phi,\ \frac{c}{2} \leqslant x_n \leqslant c,\ n \geqslant N_0\right\}, \tag{9}$$

we also define an operator $T: X \to X$ as

$$[Tx]_n = c - \sum_{s=n}^{\infty} a_s g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right), \quad n \geqslant N_0. \tag{10}$$

It is routinely verified that (i) T maps X into itself, (ii) T is a continuous mapping, (iii) $T(X)$ is precompact. In fact, if $x \in X$, then

$$\begin{aligned}c \geqslant (Tx)_n &= c - \sum_{s=n}^{\infty} a_s g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right)\\ &\geqslant c - \sum_{s=n}^{\infty} a_s g\left(H \sum_{r=s+1}^{\infty} f(r, c)\right)\\ &\geqslant \frac{c}{2}.\end{aligned}$$

Next, we show that T is continuous. Let $x^{(k)} \in X$ be such that $\lim_{k\to\infty}\|x^{(k)}\| = 0$. Since X is closed, $x \in X$ and

$$|(Tx^{(k)})_n - (Tx)_n| = \left|\sum_{s=n}^{\infty} a_s g\left(\sum_{r=s+1}^{\infty} f(r, x_r^{(k)})\right) - \sum_{s=n}^{\infty} a_s g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right)\right|$$
$$\leqslant \sum_{s=n}^{\infty} a_s \left|g\left(\sum_{r=s+1}^{\infty} f(r, x_r^{(k)})\right) - g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right)\right|.$$

By the continuity of g and f, and according to Lebesgue's dominated convergence theorem, it follows that

$$\lim_{k\to\infty} \sup_{n\geqslant N_0} |(Tx^{(k)})_n - (Tx)_n| = 0.$$

This shows that

$$\lim_{k\to\infty} \|(Tx^{(k)}) - (Tx)\| = 0.$$

Finally, we will show that TX is precompact. Let $x \in X$ and $m, n \geqslant N_0$. For $m > n$, we get

$$|(Tx)_m - (Tx)_n| = \left|\sum_{s=n}^{m-1} a_s g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right)\right|$$
$$\leqslant \left|\sum_{s=n}^{\infty} a_s g\left(H \sum_{r=s+1}^{\infty} f(r, c)\right)\right|.$$

Therefore, by the Schauder–Tychonoff fixed point theorem, there is the sequence $x = \{x_n\} \in X$ such that $x = Tx$, that is

$$x_n = c - \sum_{s=n}^{\infty} a_s g\left(\sum_{r=s+1}^{\infty} f(r, x_r)\right). \tag{11}$$

Set

$$y_n = \sum_{r=n+1}^{\infty} f(r, x_r), \quad n \geqslant N_0.$$

Then $\lim_{n\to\infty} y_n = 0$ and $\Delta y_{n-1} = -f(n, x_n)$. On the other hand,

$$x_n = c - \sum_{s=n}^{\infty} a_s g(y_n),$$

shows that $\lim_{n\to\infty} x_n = c$ and $\Delta x_n = a_n g(y_n)$. This completes the proof of the theorem. □

We see from the proof of Theorem 1, if (c_4) holds also, then we can remove the positive constant d in theorem 1, i.e. $g(u)=|u|^{\beta}sgn(u)$, $u\in R(\beta>0)$, and especially when $g(u)=u$, $u\in R$. Hence, we have

Theorem 2. *Suppose that* (c_1), (c_2) *and* (c_4) *are satisfied. Then, system* (1) *has a nonoscillatory solution* $(\{x_n\},\{y_n\})$ *such that* $\lim_{n\to\infty}x_n=\alpha\neq 0$ *and* $\lim_{n\to\infty}y_n=0$ *if and only if*

$$\sum_{n_0}^{\infty}a_n g\left(\sum_{r=n}^{\infty}|f(r,c)|\right)<\infty \quad \textit{for some } c\neq 0. \tag{12}$$

Theorem 3. *Suppose that* (c_1), (c_3) *and* (c_5) *are satisfied. Then, system* (1) *has a nonoscillatory solution* $(\{x_n\},\{y_n\})$ *such that* $\lim_{n\to\infty}(x_n/A_{n_0,n})=\alpha\neq 0$, $\lim_{n\to\infty}y_n=\beta\neq 0$ *if and only if*

$$\sum_{n_0}^{\infty}|f(n,cA_{n_0,n)}|<\infty \quad \textit{for some } c\neq 0. \tag{13}$$

Proof. Necessity. Suppose that system (1) has a solution $x=(\{x_n\},\{y_n\})$ such that $\lim_{n\to\infty}(x_n/A_{n_0,n})=$ const $\neq 0$, $\lim_{n\to\infty}y_n=\beta$, we may assume that $\lim_{n\to\infty}(x_n/A_{n_0,n})>0$, hence $\Delta y_n<0$ and $y_n>\beta$. There exist positive constants l, L and $n_1\in N(n_0)$ such that $lA_{n_0,n}\leqslant x_n\leqslant LA_{n_0,n}$ for $n\geqslant n_1$. Summing the second equation of system (1) from n_1+1 to ∞, we have

$$\sum_{r=n_1+1}^{\infty}f(r,x_r)=y_{n_1}-\beta<\infty. \tag{14}$$

By (c_2), there exists a constant $h>0$ such that $f(n,x_n)\geqslant hf(n,lA_{n_0,n})$, $n\geqslant n_1$, it follows that

$$h\sum_{r=n_1+1}^{\infty}f(r,lA_{n_0,r})<\sum_{r=n_1+1}^{\infty}f(r,x_r)<\infty,$$

which implies that (13) holds with $c=l$.

Sufficiency. Let (13) hold for some $c=2k$, where $k>0$. By condition (c_3) there exists a constant $H>0$ such that $k\leqslant x_n\leqslant 2k$ implies

$$f(n,A_{n_0,n}x_n)\leqslant Hf(n,2kA_{n_0,n}),\quad n\in N(n_0).$$

Let $N_0\in N(n_0)$ be large enough so that

$$H\sum_{n=N_0}^{\infty}f(n,2kA_{n_0,n})\leqslant w, \tag{15}$$

where $w=g^{-1}(c)/2$. We introduce the linear space Φ of all sequences $X=\{x_n\}$ such that $\sup_{n\in N(n_0)}(|x_n|/A_{N_0,n})<\infty$. Further, we define this sup to be the norm, that is, if $x\in\Phi$, then $\|x\|=\sup_{n\geqslant N_0}(|x_n|/A_{N_0,n})$. It is obvious that Φ is a Banach space with the norm. Define

$$X=\{x\in\Phi,\ g(w)A_{N_0,n}\leqslant x_n\leqslant g(2w)A_{N_0,n},\ n>N_0\}. \tag{16}$$

It is easy to see that X is a bounded, convex and closed subset of Φ, we also define an operator $T: X \to X$ by the formula

$$[Tx]_n = \sum_{s=N_0}^{n-1} a_s g\left(w + \sum_{r=s+1}^{\infty} f(r, y_r)\right), \quad n \geqslant N_0. \tag{17}$$

Since it can be shown that T is continuous and sends X into a relatively compact subset of X, the Schauder–Tychonoff fixed point theorem ensures that the existence of a sequence $x = \{x_n\} \in X$ such that $x = Tx$, that is

$$x_n = \sum_{s=N_0}^{n-1} a_s g\left(w + \sum_{r=s}^{\infty} f(r, x_r)\right), \quad n \geqslant N_0. \tag{18}$$

Set

$$y_n = w + \sum_{r=n+1}^{\infty} f(r, x_r), \quad n \geqslant N_0.$$

Then $\lim_{n\to\infty} y_n = w$ and $\Delta y_{n-1} = -f(n, x_n)$. On the other hand,

$$x_n = \sum_{s=N_0}^{n-1} a_s g\left(w + \sum_{r=s+1}^{\infty} f(r, y_r)\right).$$

By Stolz' theorem,we have $\lim_{n\to\infty}(x_n/A_{n_0,n}) = \lim_{n\to\infty} g(w + \sum_{r=n+1}^{\infty} f(r, y_r)) = g(w) \neq 0$ and $\Delta x_n = a_n g(y_n)$. This completes the proof of the theorem. □

3. Oscillation results

In this section, we need some additional conditions to guarantee that system (1) has oscillatory solutions.

(c_6) There exists a positive constant d such that $g(uv) \geqslant dg(u)g(v)$, for $uv > 0$.

(c_7) There exists a continuous nondecreasing function $\varphi : R \to R$ with the properties such that

$$sgn\,\varphi(y) = sgn\,y, \quad \int^{\pm\infty} \frac{\mathrm{d}u}{g(\varphi(u))} < \infty, \tag{19}$$

and

$$|f(n, y)| \geqslant |f(n, l)|\varphi(y)|, \quad n \in N(n_0), \quad |y| \geqslant y_0, \tag{20}$$

for some constants $y_0 > 0$ and $l \neq 0$ with $sgn\,l = sgn\,y$.

(c_8) $f(n, y)$ is nondecreasing in y for each fixed $n \geqslant n_0$.

(c_9) There exist a continuous nondecreasing function $\varphi : [-M, M] \to R$, $M > 0$ being a constant, such that

$$sgn\,\varphi(v) = sgn\,v \quad \text{and} \quad \int_{\pm 0}^{\pm M} \frac{\mathrm{d}v}{\varphi(g(v))} < \infty, \tag{21}$$

and

$$|f(n,uv)| \geqslant k|f(n,u)|\varphi(v)|, \quad n \in N(n_0), \quad u \neq 0, \quad 0 < |v| < v_0, \tag{22}$$

for some positive constant $k > 0$ and $v_0 > 0$.

Theorem 4. *Suppose that* (c_1), (c_2), (c_5)–(c_7) *is satisfied. Then, all solutions of system* (1) *are oscillatory if and only if*

$$\sum_{n_0}^{\infty} a_n g\left(\sum_{r=n+1}^{\infty} |f(r,c)|\right) = \infty \quad \textit{for all } c \neq 0. \tag{23}$$

Proof. Necessity. If (23) is violated, then, by Theorem 1, system (1) has a nonoscillatory solution $(\{x_n\}, \{y_n\})$ such that $\lim_{n\to\infty} x_n = \text{const} \neq 0$ and $\lim_{n\to\infty} y_n = 0$.

Sufficiency. Let (23) hold and suppose that (1) has a nonoscillatory solution $(\{x_n\}, \{y_n\})$. We may assume that $x_n > 0$ for $n \geqslant n_1$. In view of (c_5), similar to the proof of Lemma 4 [9], we get $y_n > 0$, it follows from the first equation of system (1) that $\Delta x_n > 0$, $n \geqslant n_1$, and $\lim_{n\to\infty} x_n = \infty$ by Theorem 1. From the second equation of system (1), we have $\Delta y_n < 0$, hence, $\lim_{n\to\infty} y_n \geqslant 0$.

Summing the second equation of (1) from $n+1$ to ∞, we have

$$y_n \geqslant \left(\sum_{r=n+1}^{\infty} f(r, y_r)\right), \quad n \geqslant n_1, \tag{24}$$

by (24), (c_6) and in view of nondecreasing φ, it follows that

$$\frac{\Delta x_n}{g(\varphi(x_{n+1}))} = \frac{a_n g(y_n)}{g(\varphi(x_{n+1}))} \geqslant \frac{a_n g(\sum_{r=n+1}^{\infty} f(r, x_r))}{g(\varphi(x_{n+1}))} \geqslant d a_n g\left(\sum_{r=n+1}^{\infty} \frac{f(r, x_r)}{\varphi(x_r)}\right), \quad n \geqslant n_1. \tag{25}$$

Since $x_n \to \infty$ as $n \to \infty$, we see in view of (20) that there is $n_2 \geqslant n_1$ such that

$$\frac{f(n, x_n)}{\varphi(y_n)} \geqslant f(n, l), \quad n \geqslant n_2, \tag{26}$$

for $l > 0$. From (25) and (26), we have

$$\frac{\Delta x_n}{g(\varphi(x_{n+1}))} \geqslant d a_n g\left(\sum_{r=n+1}^{\infty} f(r, l)\right). \tag{27}$$

Summing two sides of (27) from n_2 to n, we obtain

$$\sum_{r=n_2}^{n} \frac{\Delta x_r}{g(\varphi(x_{r+1}))} \geqslant \sum_{s=n_2}^{n} d a_s g\left(\sum_{r=s+1}^{\infty} f(r, l)\right). \tag{28}$$

Define

$$r(t) = x_n + (t-n)\Delta x_n, \quad n \leqslant t \leqslant n+1.$$

In view of $\Delta x_n \geqslant 0$, then $x_n \leqslant r(t) \leqslant x_{n+1}$ and

$$\frac{\Delta x_n}{g(\varphi(x_{n+1}))} \leqslant \frac{r'(t)}{g(\varphi(r(t)))} \leqslant \frac{\Delta x_n}{g(\varphi(x_n))}. \tag{29}$$

From (23) and (24), we obtain

$$\int_{r(n_2)}^{\infty} \frac{\mathrm{d}s}{g(\varphi(s))} \geqslant \int_{n_2}^{n+1} \frac{\mathrm{d}r(t)}{g(\varphi(r(t)))} \geqslant \sum_{s=n_2}^{n} da_s g\left(\sum_{r=s+1}^{\infty} f(r,l)\right). \tag{30}$$

In view of (19) and (23), this inequality yields a contradiction and completes the proof of the theorem. □

Theorem 5. *Suppose that* (c_5), (c_8) *and* (c_9) *are satisfied. Then, all solutions of system* (1) *are oscillatory if and only if*

$$\sum_{m}^{\infty} |f(n, cA_{n_0,n})| = \infty \quad \textit{for all } c \neq 0. \tag{31}$$

Proof. Necessity. If (31) is violated, then, by Theorem 3, there is a nonoscillatory solution $(\{x_n\}, \{y_n\})$ of the system (1) such that

$$\lim_{n\to\infty} \frac{x_n}{A_{n_0,n}} = \text{const} \neq 0.$$

Sufficiency. Let (31) hold and suppose that (1) has a nonoscillatory solution $(\{x_n\}, \{y_n\})$. We may assume that $x_n > 0$ for $n \geqslant n_1$. Then by same argument in the proof of Theorem 4, we have $\Delta x_n > 0$, $\Delta y_n < 0$, $y_n > 0$ eventually. We claim that (31) implies y_n decrease to 0 as $n \to \infty$. In fact, if $\lim_{n\to\infty} = \beta > 0$, then $y_n \geqslant \beta$ for $n \geqslant n_1$. According to the first equation, we have $x_n = x_{n_1} + \sum_{s=n_1}^{n-1} a_s g(y_s) > g(\beta) A_{n_1,n}$. Summing the second equation of (1) from $n_1 + 1$ to ∞, note (c_8) and (31), we have

$$\beta - y_{n_1} = -\sum_{s=n_1+1}^{\infty} f(s, x_s) \leqslant -\sum_{s=n_1+1}^{\infty} f(s, g(\beta) A_{n_1,s}) = -\infty,$$

which is a contradiction. Hence, y_n decreases to 0 as $n \to \infty$. Summing the first equation of (1), we have

$$x_n - x_{n_1} = \sum_{s=n_1}^{n-1} a_s g(y_s) \geqslant g(y_{n-1}) A_{n_1,n}.$$

It follows that there exists a positive constant $c > 1$ such that

$$\frac{x_n}{g(y_n)} \geqslant c A_{n_1,n}. \tag{32}$$

In view of (c_5) and $g(y_n) \neq 0$, using (c_8) and (c_9), we get

$$kf(n, cA_{n_1,n}) \leqslant kf\left(n, \frac{x_n}{g(y_n)}\right) \leqslant \frac{f(n, x_n)}{\varphi(g(y_n))} = -\frac{\Delta y_{n-1}}{\varphi(g(y_n))}. \tag{33}$$

From (32), it follows

$$\sum_{s=n_0}^{n} kf(s, cA_{n_1,s}) \leqslant -\sum_{s=n_0}^{n} \frac{\Delta y_{s-1}}{\varphi(g(y_s))}. \tag{34}$$

Define

$$r(t) = y_n + (t-n)\Delta y_n, \quad n \leqslant t \leqslant n+1.$$

In view of $\Delta y_n \leqslant 0$, then $y_{n+1} \leqslant r(t) \leqslant y_n$ and

$$\frac{\Delta y_n}{\varphi(g(y_{n+1}))} \leqslant \frac{r'(t)}{\varphi(g(r(t)))} \leqslant \frac{\Delta y_n}{\varphi(g(y_n))}. \tag{35}$$

From (34) and (35), we obtain

$$\sum_{s=n_0}^{n} kf(s, cA_{n_1,s}) \leqslant \int_{r(n_0)}^{r(n)} \frac{\mathrm{d}s}{\varphi(g(s))}. \tag{36}$$

In view of (21) and (31), this is a contradiction and the proof of Theorem 5 is complete.

4. Some examples

Example 1. Consider the nonlinear difference system

$$\Delta x_n = |y_n|^{(1/\alpha)-1} y_n, \quad \Delta y_{n-1} = -\frac{n^v |x_n|^{r-1} x_n}{1 + n^u |x_n|^m}, \quad r > m, \quad n \in N = \{1, 2, \ldots\}, \tag{37}$$

where $m > 0$, $r > 0$, $u > 0$, $\alpha > 0$, and v are constants. Here

$$a_n = 1, \quad g(x) = |x|^{(1/\alpha)-1} x, \quad f(n, y) = \frac{n^v |y|^{r-1} y}{1 + n^u |y|^m}.$$

Since $0 < l \leqslant y \leqslant L$ implies

$$f(n, l) \leqslant f(n, y) \leqslant f(n, L) \quad \text{for } r \geqslant m,$$

$$f(n, ln) \leqslant f(n, yn) \leqslant f(n, Ln) \quad \text{for } r \geqslant m$$

and $g(x) = |x|^{(1/\alpha)-1} x$ is increasing. As is easily verified, (6) or (12) holds if and only if $u > \alpha + v + 1$, (13) holds if and only if $u + m > v + n + 1$. By Theorem 1 (Theorem 2), (37) has a nonoscillatory solution $(\{x_n\}, \{y_n\})$ such that $\lim_{n\to\infty} x_n \neq 0$, $\lim_{n\to\infty} y_n = 0$ if and only if $u > \alpha + v + 1$. By Theorem 3, (37) has a nonoscillatory solution $(\{x_n\}, \{y_n\})$ such that $\lim_{n\to\infty}(x_n/n) \neq 0$, $\lim_{n\to\infty} y_n \neq 0$ if and only if $u + m > v + n + 1$.

Example 2. Consider the nonlinear difference system

$$\Delta x_n = \frac{1}{n} y_n^4, \quad \Delta y_{n-1} = -\frac{n^3 |x_n|^{4/3} x_n}{2004 + n^3 x_n^2}, \quad n \in N = \{1, 2, \ldots\}, \tag{38}$$

where

$$a_n = 1/n, \quad g(x) = x^4, \quad f(n, y) = \frac{n^3|y|^{4/3}y}{2004 + n^3y^2}.$$

Obviously, $f(n, y)$ is increasing in y for fixed n, and taking $\varphi(u) = |u|^{-2/3}u$, we have

$$|f(n, y)| \geqslant |f(n, sgn\, y)||y|^{1/3} \quad \text{for } |y| \geqslant 1. \tag{39}$$

and

$$\int^{\pm\infty} \frac{\mathrm{d}u}{g(\varphi(u))} = \int^{\pm\infty} \frac{\mathrm{d}u}{u^{4/3}} < \infty. \tag{40}$$

It is easy to see (23) holds, so the conditions of Theorem 4 are satisfied. Hence all solutions of (38) one oscillatory.

Example 3. Consider the nonlinear difference equation

$$\Delta^2 x_{n-1} + p_n|x_n|^{\lambda} sgn\, x_n = 0. \tag{41}$$

In [5], authors proved the following results for (41) .

(i) If $\lambda > 1$, then a necessary and sufficient condition for oscillation is

$$\sum_{n=n_0}^{\infty} np_n = \infty. \tag{42}$$

(ii) If $0 < \lambda < 1$, then a necessary and sufficient condition for oscillation is

$$\sum_{n=n_0}^{\infty} n^{\lambda} p_n = \infty. \tag{43}$$

(41) can be written in the form:

$$\Delta x_n = y_n, \quad \Delta y_{n-1} = p_n|x_n|^{\lambda} sgn\, x_n, \quad n \in N(n_0) = \{n_0, n_0 + 1, \ldots\}, \tag{44}$$

where $a_n = 1$, $g(x) = x$, $f(n, y) = p_n|x_n|^{\lambda} sgn\, x_n$.

If $\lambda > 1$, we take $\varphi(x) = |x|^{\lambda-1}x$, $|x| \geqslant 1$, then all conditions of Theorem 4 are satisfied. Hence (41) is oscillatory if and only if

$$\sum_{n=n_0}^{\infty} \sum_{r=n+1}^{\infty} p_r = \infty. \tag{45}$$

By interchanging the order of summing, it is equivalent to (41).

If $0 < \lambda < 1$, we take $\varphi(x) = |x|^{\lambda-1}x$, $0 < |x| \leqslant 1$, then all conditions of Theorem 5 are satisfied.

Hence (41) is oscillatory if and only if

$$\sum_{n=n_0}^{\infty} n^{\lambda} p_n = \infty. \tag{46}$$

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